ON THE EXISTENCE OF STABLE COMPACT LEAVES FOR TRANSVERSELY HOLOMORPHIC FOLIATIONS

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1. Introduction

One of the most important results in the theory of foliations is the celebrated Local Stability Theorem of Reeb (see for instance [5, 9]): *A compact leaf of a foliation having finite holonomy group is stable, indeed, it admits a fundamental system of invariant neighborhoods where each leaf is compact with finite holonomy group*[1]. This result, together with Reeb's Global Stability Theorem (for codimension one real foliations) has many important consequences and motivates several questions in the theory of foliations. Recall that a leaf L of a *compact* foliation $\mathcal{F}$ is *stable* if it has a fundamental system of saturated neighborhoods (cf.[5] page 376). The stability of a compact leaf $L \in \mathcal{F}$ is equivalent to finiteness of its holonomy group $\mathrm{Hol}(\mathcal{F}, L)$ and is also equivalent to the existence of a local bound for the volume of the leaves close to L ([5], Proposition 2.20, page 103). As a converse to the above, Reeb has proved that, for codimension one smooth foliations, a compact leaf that admits a neighborhood consisting of compact leaves, is necessarily of finite holonomy. This is not true however in codimension ≥ 2.

Some interesting questions arise from these deep results. One is the following: *If a codimension one smooth foliation on a compact manifold has infinitely many compact leaves then is it true that all leaves are compact?* The answer is clearly no, but this is true for (transversely) real analytic foliations of codimension one on compact manifolds. On the other, there are versions of Reeb stability results for the class of *holomorphic* foliations ([1, 8]). In the holomorphic framework, it is proved in [2] that a (non-singular) transversely holomorphic codimension one on a compact connected manifold admitting infinitely many compact leaves exhibits a transversely meromorphic first integral. In [6] the author proves a similar result, if a (possibly singular) codimension one holomorphic foliations on a compact manifold has infinitely many closed leaves then it admits a meromorphic first integral and in particular, all leaves are closed. The problem of bounding the number of closed leaves of a holomorphic foliation is known (at least in the complex algebraic framework) as *Jouanolou's problem*, thanks to the pioneering results in [7] and has a wide range of contributions and applications in the Algebraic-geometric setting.

From the more geometrical point of view, some interesting questions arise from the above results. In [8] it is proved a global stability theorem for codimension $q \geq 1$ holomorphic foliations transverse to fibrations. In this paper we focus on the problem of existence of a suitable compact leaf under the hypothesis of existence of a sufficiently large number of compact leaves.

We recall that a subset $X \subset M$ of a differentiable m-manifold has *zero measure on M* if M admits an open cover by coordinate charts $\varphi\colon U \subset M \to \varphi(U) \subset \mathbb{R}^m$ such that $\varphi(U \cap X)$ has zero

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measure (with respect to the standard Lebesgue measure in $\mathbb{R}^m$. Our results are stated in terms of positive measure and we prove the following theorems:

Theorem 1.1. *Let $\mathcal{F}$ be a transversely holomorphic foliation on a compact connected complex manifold M. Denote by $\Omega(\mathcal{F}) \subset M$ the subset of compact leaves of $\mathcal{F}$. Then we have two possibilities:*

(i) *$\mathcal{F}$ has some compact leaf with finite holonomy group.*
(ii) *The set $\Omega(\mathcal{F})$ has zero measure.*

A compact leaf with finite holonomy will be called *stable*. In view of the Reeb local stability theorem ([4, 5, 9], a stable leaf always belongs to the interior of the set of compact leaves, therefore Theorem 1.1 can be stated as:

A transversely holomorphic foliation on a compact complex manifold, exhibits a compact stable leaf if and only if the set of compact leaves is not a zero measure subset of the manifold. Parallel to this result we have the following version for groups:

Theorem 1.2. *Let $G \subset \text{Diff}(F)$ be a subgroup of holomorphic diffeomorphisms of a complex connected manifold F. Denote by $\Omega(G)$ the subset of points $x \in F$ such that the G-orbit of x is periodic. There are two possibilities:*

(i) *G is a finite group.*
(ii) *The set $\Omega(G)$ has zero measure on F.*

We point-out that the subgroup $G \subset \text{Diff}(F)$ is not supposed to be finitely generated.

2. Holonomy and stability

Let $\mathcal{F}$ be a codimension k holomorphic foliation on a complex manifold M. Given a point $p \in M$, the leaf through p is denoted by L_p. We denote by $\text{Hol}(\mathcal{F}, L_p) = \text{Hol}(L_p)$ the holonomy group of L_p. This is a conjugacy class of equivalence, and we shall denote by $\text{Hol}(L_p, \Sigma_p, p)$ its representative given by the local representation of this holonomy calculated with respect to a local transverse section Σ_p centered at the point $p \in L_p$. The group $\text{Hol}(L_p, \Sigma_p, p)$ is therefore a subgroup of the group of germs $\text{Diff}(\Sigma_p, p)$ which is identified with the group $\text{Diff}(\mathbb{C}^k, 0)$ of germs at the origin $0 \in \mathbb{C}^k$ of complex diffeomorphisms.

The classical Reeb local stability theorem ([4, 5]) states that if L_0 is a compact leaf with finite holonomy of a smooth foliation $\mathcal{F}$ on a manifold M then there is a fundamental system of invariant neighborhoods W of L_0 in $\mathcal{F}$ such that every leaf $L \subset W$ is compact, has a finite holonomy group and admits a finite covering onto L_0. Moreover, for each neighborhood W of L_0 there is an $\mathcal{F}$-invariant tubular neighborhood $\pi\colon W' \subset W \to F$ of F with the following properties:

(1) Every leaf $L' \subset W'$ is compact with finite holonomy group.
(2) If $L' \subset W'$ is a leaf then the restriction $\pi\big|_{L'}\colon L' \to L$ is a finite covering map.
(3) If $x \in L$ then $\pi^{-1}(x)$ is a transverse of $\mathcal{F}$.
(4) There is an uniform bound $k \in \mathbb{N}$ such that for each leaf $L \subset W$ we have $\#(L \cap \pi^{-1}(x)) \leq k$.

3. Periodic groups and groups of finite exponent

Next we present Burnside's and Schur's results on periodic linear groups. Let G be a group with identity $e_G \in G$. The group is *periodic* if each element of G has finite order. A periodic group G is *periodic of bounded exponent* if there is an uniform upper bound for the orders of its elements. This is equivalent to the existence of $m \in \mathbb{N}$ with $g^m = 1$ for all $g \in G$ (cf. [8]). Because of this, a

group which is periodic of bounded exponent is also called a group of *finite exponent*. The following classical results are due to Burnside and Schur.

Theorem 3.1 (Burnside, 1905 [3], Schur, 1911 [10])**.** *Let $G \subset \mathrm{GL}(k, \mathbb{C})$ be a complex linear group.*

(i) (Burnside) *If G is of finite exponent ℓ (but not necessarily finitely generated) then G is finite; actually we have $|G| \leq \ell^{k^2}$.*

(ii) (Schur) *If G is finitely generated and periodic (not necessarily of bounded exponent) then G is finite.*

Using these results we may prove:

Lemma 3.2 ([8])**.** *About periodic groups of germs of complex diffeomorphisms we have:*

(1) *A finitely generated periodic subgroup $G \subset \mathrm{Diff}(\mathbb{C}^k, 0)$ is necessarily finite. A (not necessarily finitely generated) subgroup $G \subset \mathrm{Diff}(\mathbb{C}^k, 0)$ of finite exponent is necessarily finite.*

(2) *Let $G \subset \mathrm{Diff}(\mathbb{C}^k, 0)$ be a finitely generated subgroup. Assume that there is an invariant connected neighborhood W of the origin in $\mathbb{C}^k$ such that each point x is periodic for each element $g \in G$. Then G is a finite group.*

(3) *Let $G \subset \mathrm{Diff}(\mathbb{C}^k, 0)$ be a (not necessarily finitely generated) subgroup such that for each point x close enough to the origin, the pseudo-orbit of x is periodic of (uniformly bounded) order $\leq \ell$ for some $\ell \in \mathbb{N}$, then G is finite.*

Proof. We first prove (1). Let G be a not necessarily finitely generated subgroup of $\mathrm{Diff}(\mathbb{C}^k, 0)$, with finite exponent. We consider the homomorphism $D\colon \mathrm{Diff}(\mathbb{C}^k, 0) \to \mathrm{GL}(k, \mathbb{C})$ given by the derivative $Dg := g'(0)$, $g \in G$. Then the image DG is isomorphic to the quotient $G/\operatorname{Ker}(D)$ where the kernel $\operatorname{Ker}(D)$ is the group $G_1 = \{g \in G,\ g'(0) = \mathrm{Id}\}$, i.e., the normal subgroup of elements tangent to the identity. Since G is of finite exponent the same holds for DG as a consequence of the Chain-Rule. By Burnside's theorem above DG is a finite group. Let us now prove that G_1 is trivial. Indeed, take an element $h \in G_1$. Since h has finite order it is analytically linearizable and therefore $h = \mathrm{Id}$. Now we assume that $G \subset \mathrm{Diff}(\mathbb{C}^k, 0)$ is finitely generated and periodic. Again we consider the homomorphism $D\colon \mathrm{Diff}(\mathbb{C}^k, 0) \to \mathrm{GL}(k, \mathbb{C})$ and the image $DG \simeq G/G_1$ as above. Since G is finitely generated and periodic the same holds for DG as a consequence of the Chain-Rule. By Schur's theorem above DG is a finite group. As above G_1 is trivial and therefore G is finite.

Now we prove (2). Fix an element $g \in G$. For each $k \in \mathbb{N}$ define $X_k := \{x \in W, g^k(x) = x\}$. We claim that X_k is a closed subset of W: indeed, if $x_\nu \in X_k$ is a sequence of points converging to a point $a \in W$ then clearly $g^k(a) = a$ and therefore $a \in X_k$. By the Category theorem of Baire there is $k \in \mathbb{N}$ such that X_k has non-empty interior and therefore by the Identity theorem we have $g^k = \mathrm{Id}$ in W. This shows that each element $g \in G$ is periodic. Since G is finitely generated this implies by (1) that G is finite. The proof (3) is pretty similar to this. □

Given a subgroup $G \subset \mathrm{Diff}(F)$ and a point $p \in F$ the *stabilizer* of p in G is the subgroup $G(p) \subset G$ of elements $f \in G$ such that $f(p) = 0$. From the above we immediately have:

Proposition 3.3. *Let $G \subset \mathrm{Diff}(F)$ be a (not necessarily finitely generated) subgroup of holomorphic diffeomorphisms of a connected complex manifold F. If G is periodic and finitely generated or G is periodic of finite exponent, then each stabilizer subgroup of G is finite.*

The following simple remark gives the finiteness of finite exponent groups of holomorphic diffeomorphisms having a periodic orbit.

Proposition 3.4 (Finiteness lemma)**.** *Let G be a subgroup of holomorphic diffeomorphisms of a connected complex manifold F. Assume that:*

(1) *G is periodic of finite exponent or G is finitely generated and periodic.*
(2) *G has a finite orbit in F.*

Then G is finite.

Proof. Fixed a point $x \in F$ with finite orbit we can write $\mathcal{O}_G(x) = \{x_1, ..., x_k\}$ with $x_i \neq x_j$ if $i \neq j$. Given any diffeomorphism $f \in G$ we have $\mathcal{O}_G(f(x)) = \mathcal{O}_G(x)$ so that there exists an unique element $\sigma \in S_k$ of the symmetric group such that $f(x_j) = x_{\sigma_f(j)}$, $\forall j = 1, ..., k$. We can therefore define a map

$$\eta\colon G \to S_k,\ \eta(f) = \sigma_f\,.$$

Now, if $f, g \in G$ are such that $\eta(f) = \eta(g)$, then $f(x_j) = g(x_j)$, $\forall j$ and therefore $h = f\,g^{-1} \in G$ fixes the points $x_1, ..., x_k$. In particular h belongs to the stabilizer G_x. By Proposition 3.3 (1) and (2) (according to G is finitely generated or not) the group G_x is finite. Thus, the map $\eta\colon G \to S_k$ is a finite map. Since S_k is a finite group this implies that G is finite as well.

□

4. Measure and finiteness

Let us now prove Theorems 1.1 and 1.2. For sake of simplicity we will adopt the following notation: if a subset $X \subset M$ is not a zero measure subset then we shall write $\mu(X) > 0$. This may cause no confusion for we are not considering any specific measure μ on M and we shall be dealing only with the notion of zero measure subset. Nevertheless, we notice that if $X \subset M$ writes as a countable union $X = \bigcup\limits_{n\in\mathbb{N}} X_n$ of subsets $X_n \subset M$ then X has zero measure in M if and only if X_n has zero measure in M for *all* $n \in \mathbb{N}$. In terms of our notation we have therefore $\mu(X) > 0$ if and only if $\mu(X_n) > 0$ for *some* $n \in \mathbb{N}$.

Proof of Theorem 1.1. Because M is compact there is a finite number of relatively compact open disks $T_j \subset M$, $j = 1, ..., r$ such:

(1) Each T_j is transverse to $\mathcal{F}$ and the closure $\overline{T_j}$ is contained in the interior of a transverse disc Σ_j to $\mathcal{F}$.
(2) Each leaf of $\mathcal{F}$ intersects at least one of the disks T_j.

Put $T = \bigcup\limits_{j=1}^{r} T_j \subset M$ and define

$$\Omega(\mathcal{F}, T) = \{L \in \mathcal{F} : \#(L \cap T) < \infty\}.$$

Then $\Omega(\mathcal{F}, T) = \bigcup\limits_{n=1}^{\infty} \Omega(\mathcal{F}, T, n)$ where

$$\Omega(\mathcal{F}, T, n) = \{L \in \mathcal{F} : \#(L \cap T) \leq n\}.$$

Claim 4.1. *We have $\Omega(\mathcal{F}) = \Omega(\mathcal{F}, T)$.*

Proof. Indeed, given a leaf $L \in \mathcal{F}$ if $L \notin \Omega(\mathcal{F}, T)$ then there is some j such that $\#(L \cap T_j) = \infty$. Since $\overline{T_j}$ is compact there is a point $p \in \Sigma_j$ belonging to the closure of L and which is accumulated by points in L. Since $p \in \Sigma_j$ which is transverse to $\mathcal{F}$ we conclude that L has infinitely many plaques intersecting any distinguished neighborhood of p in M and therefore L cannot be compact.

Conversely, suppose that $L \in \Omega(\mathcal{F}, T)$ then L has only finitely many plaques in a (finite) covering of M by distinguished neighborhoods. Since M is compact this implies that L is compact. $\square$

Because $\Omega(\mathcal{F}) = \Omega(\mathcal{F}, T) = \bigcup\limits_{n \in \mathbb{N}} \Omega(\mathcal{F}, T, n)$ and $\mu(\Omega(\mathcal{F})) > 0$, there is $n \in \mathbb{N}$ such that $\mu(\Omega(\mathcal{F}, T, n)) > 0$.

Next we claim:

Claim 4.2. *There is are a compact leaf $L_0 \in \Omega(\mathcal{F}, T)$ and fundamental system of open neighborhoods V of L_0 in M such that*

$$\mu(\Omega(\mathcal{F}, T, n) \cap V) > 0.$$

Proof. Indeed, otherwise for each compact leaf $L \in \Omega(\mathcal{F})$ and for each neighborhood V_L of L in M there is a neighborhood $W_L \subset V_L$ of L in M such that $\mu(W_L \cap \Omega(\mathcal{F}, T, n)) = 0$. In particular there is an open cover $\Omega(\mathcal{F}, T, n) \subset \bigcup\limits_{L \in \Omega(\mathcal{F}, T, n)} W_L$ such that $\mu(W_L \cap \Omega(\mathcal{F}, T, n)) = 0$. The open cover admits a countable subcover so that we have $\Omega(\mathcal{F}, T, n) \subset \bigcup\limits_{n \in \mathbb{N}} W_n$ with $\mu(W_n) = 0$, $\forall n \in \mathbb{N}$. This implies $\mu(\Omega(\mathcal{F}, T, n)) = 0$, a contradiction. $\square$

Let therefore $L_0 \in \Omega(\mathcal{F}, T, n)$ be as above. We may choose a base point $p \in L_0 \cap T$ and a transverse disc $\Sigma_p \subset \overline{\Sigma_p} \subset T$ to $\mathcal{F}$ centered at p. Given a point $z \in \Sigma_p$ we denote the leaf through z by L_z. If $L_z \in \Omega(\mathcal{F}, T, n)$ then $\#(L_z \cap \Sigma_p) \le n$.

Take now a holonomy map germ $h \in \mathrm{Hol}(\mathcal{F}, L_0, \Sigma_p, p)$. Let us choose a sufficiently small subdisk $W \subset \Sigma_p$ such that the germ h has a representative $h \colon W \to \Sigma_p$ such that the iterates $h, h^2, ..., h^{n+1}$ are defined in W. Because of the claim above we have $\mu(\{z \in W : \# L_z \cap \Sigma_p \le n\}) > 0$.

Put $X =: \{z \in W : \# L_z \cap \Sigma \le n\}$. Given a point $z \in X$ we have $h^\ell(z) = z$ for some $\ell \le n$. Therefore there is $n_h \le n$ such that

$$\mu(\{z \in W : h^{n_h}(z) = z\}) > 0$$

Since h is analytic, the set $\{z \in W : h^{n_h}(z) = z\}$ is an analytic subset of W and therefore either this coincides with W or (it has codimension ≥ 2 and therefore) this is a zero measure subset of W. We conclude that $h^{n_h} = \mathrm{Id}$ in W. This shows that each germ $h \in \mathrm{Hol}(\mathcal{F}, L_0, \Sigma_p, p)$ is periodic of order $n_h \le n$ for some uniform $n \in \mathbb{N}$. This implies that this holonomy group is finite by Proposition 3.4. $\square$

Proof of Theorem 1.2. Thanks to Burnside's theorem (3.1) and and Proposition 3.4 it is enough to prove the following claim:

Claim 4.3. *G is a periodic group of finite exponent.*

proof of the claim. We have $\Omega(G) = \{x \in F : \#\mathcal{O}_G(x) < \infty\} = \bigcup\limits_{k=1}^{\infty} \{x \in F : \#\mathcal{O}_G(x) \le k\}$, therefore there is some $k \in \mathbb{N}$ such that

$$\mu(\{x \in F : \#\mathcal{O}_G(x) \le k\}) > 0.$$

In particular, given any diffeomorphism $f \in G$ we have

$$\mu(\{x \in F : \#\mathcal{O}_f(x) \le k\}) > 0.$$

Therefore, there is $k_f \leq k$ such that the set $X = \{x \in F : f^{k_f}(x) = x\}$ has positive measure. Since $X \subset F$ is an analytic subset, this implies that $X = F$ (a proper analytic subset of a connected complex manifold has (codimension ≥ 2 and therefore it has) zero measure in F). Therefore, we have $f^{k_f} = \mathrm{Id}$ in F. This shows that G is periodic of finite exponent. $\square$

$\square$

References

[1] Brunella, Marco. *A global stability theorem for transversely holomorphic foliations.* Ann. Global Anal. Geom. 15 (1997), no. 2, 179–186.

[2] Brunella, Marco; Nicolau, Marcel. Sur les hypersurfaces solutions des équations de Pfaff. C. R. Acad. Sci. Paris Sér. I Math. 329 (1999), no. 9, 793–795.

[3] Burnside, W.: *On criteria for the finiteness of the order of a group of linear substitutions*, Proc.London Math. Soc. (2) 3 (1905), 435-440.

[4] Camacho, César; Lins Neto, Alcides. *Geometry theory of foliations.* Translated from the Portuguese by Sue E. Goodman. Birkhäuser Boston, Inc., Boston, MA, 1985. vi + 205 pp.

[5] Godbillon, Claude. *Feuilletages* (French) [Foliations] Études géométriques [Geometric studies]. With a preface by G. Reeb. Progress in Mathematics, 98. Birkhäuser Verlag, Basel, 1991. xiv + 474 pp.

[6] Ghys, Étienne: À propos d'un théorème de J.-P. Jouanolou concernant les feuilles fermées des feuilletages holomorphes. (French. English summary) [On a theorem of J.-P. Jouanolou concerning the closed leaves of holomorphic foliations] Rend. Circ. Mat. Palermo (2) 49 (2000), no. 1, 175–180.

[7] Jouanolou, Jean-Pierre. *Équations de Pfaff algèbriques*; Lecture Notes in Math. 708, Springer-Verlag, Berlin, 1979.

[8] Santos, Fábio; Scardua, Bruno. *Stability of complex foliations transverse to fibrations*, to appear in Proceedings of the American Mathematical Society.

[9] Reeb, Georges. *Variétés feuilletées, feuilles voisines*; C.R.A.S. Paris 224 (1947), 1613-1614.

[10] Schur, I.: *Über Gruppen periodischer substitutionen*, Sitzungsber. Preuss. Akad. Wiss. (1911), 619–627.

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